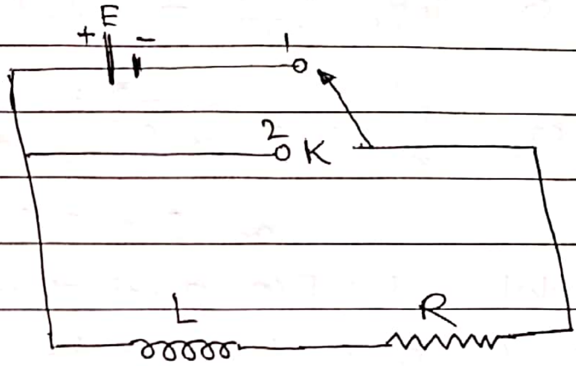
Growth of current in L-R circuit (Helmholtz Equation)

Consider a circuit having an inductance L and a resistance R placed in series with a battery of emf E as shown in fig (1).

When the key K is in position 1, the current slowly increases from zero to a maximum value I_0 after some time. During the time the current is growing there is a back emf in the inductance.

Let I be the value of the current at any instant during the variable state and $\frac{dI}{dt}$ be the rate at which the current grows, then



Fall of potential across the resistance $= RI$

Back emf in the inductance

$$= -L\frac{dI}{dt}$$

Since Back emf opposes the emf E due to the battery. Thus with two sets of emf the net emf is $E - L\frac{dI}{dt}$.

Applying Kirchhoff's 2nd Law (or voltage Law) Now gives the loop equation as

$$\left[E - L\frac{dI}{dt}\right] = RI$$

Dividing both side by R, we have

$$\frac{E}{R} - \frac{L\,dI}{R\,dt} = I$$

or $$\frac{E}{R} - I = \frac{L}{R}\frac{dI}{dt}$$ Dividing both Sides by dI

and taking the reciprocals, we have $$\frac{dI}{E/R - I} = \frac{R}{L}\,dt$$

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or $$\frac{-dI}{\frac{E}{R}-I} = \frac{R}{L}dt$$

Integrating both sides, we have

$$\log_e\left(\frac{E}{R}-I\right) = -\frac{R}{L}t + C \quad \text{—— (1)}$$

Where C is constant of integration

When $t=0$, Substituting these values in eqn (1), we have

$$\log_e\left(\frac{E}{R}\right) = C \quad \therefore \log_e\left(\frac{E}{R}-I\right) = \log_e\left(\frac{E}{R}\right) - \frac{R}{L}t$$

or $\log_e \frac{E/R - I}{E/R} = -\frac{R}{L}t$ or $\frac{E/R - I}{E/R} = e^{-\frac{R}{L}t}$

or $\frac{E}{R} - I = \frac{E}{R}e^{-\frac{R}{L}t}$ or $I = \frac{E}{R}\left(1-e^{-\frac{R}{L}t}\right)$ —— (2)

or $I = I_0\left(1-e^{-\frac{R}{L}t}\right)$ —— (3)

Where $I_0 = E/R$ gives final steady value of the current

eqn (3) gives the growth of current in L-R circuit.

Conclusion: (i) At $t=0$, $I=0$

(ii) As t increases, the 2nd term decreases and hence current increases expontially.

(iii) At $t \to \infty$, the ~~current~~ tends to steady state value $I_0 = E/R$.

The curve of current I against t is

Time constant: The fraction L/R is called the time constant of the circuit

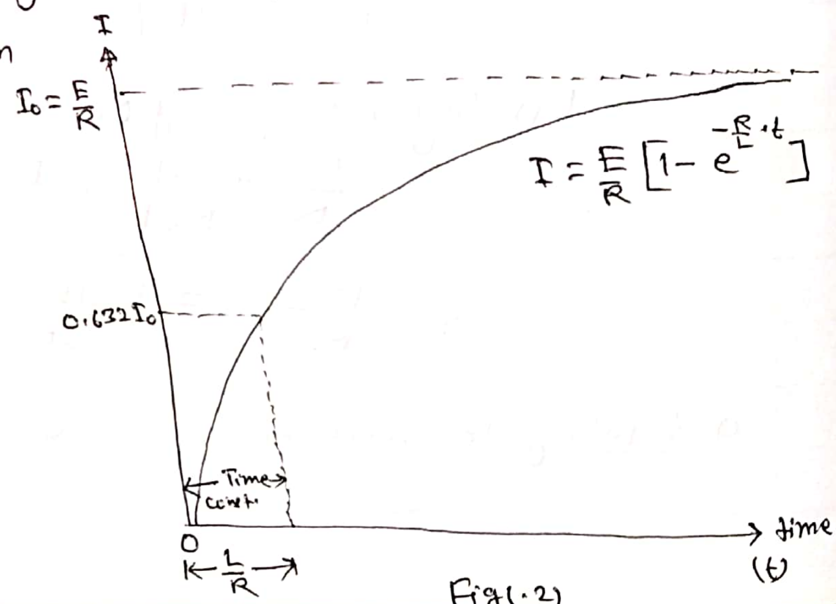
If $t = \frac{L}{R}$, then

$$I = I_0\left(1-e^{-t/t}\right) = I_0\left(1-e^{-1}\right)$$

$$= I_0\left(1-\frac{1}{e}\right) = I_0\left(\frac{e-1}{e}\right)$$

$$= I_0\left(\frac{1.718}{2.718}\right)$$

$$\therefore I = 0.632\, I_0$$



Fig(.2)

Hence time constant of LR circuit is defined as time during which the current rise to be maxm

Decay of current in L-R circuit

After the steady current has been established, the source of steady emf is now removed and connecting to position 2, the current begins to decay (fall) exponentially in L-R circuit as show in fig (2).

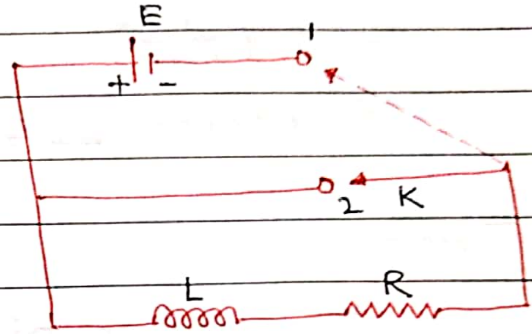


Fig.(2)

During the time the current is decreasing, there is an emf. induced in the inductance having a value $e = -L\frac{dI}{dt}$. The differential loop equation using Kirchhoff's 2nd Law as

$$\left[0 - L\frac{dI}{dt}\right] = RI$$

or $\frac{dI}{I} = -\frac{R}{L}t$ Integrating both sides, we get

$$\int \frac{dI}{I} = -\frac{R}{L}\int dt + B$$

where B is a constant of integration.

$$\therefore \log_e I = -\frac{R}{L}t + B \quad \text{------ (4)}$$

when $t = 0$, then $I = I_0$

$$\therefore \log_e I_0 = 0 + B \quad \text{or} \quad B = \log_e I_0$$

Substituting for B in equ^n (4), we get

$$\log_e I = -\frac{R}{L}t + \log_e I_0$$

or $\log_e I - \log_e I_0 = -\frac{R}{L}t$

or $\log_e\left(\frac{I}{I_0}\right) = e^{-\frac{R}{L}t}$

or $\frac{I}{I_0} = e^{-\frac{R}{L}t}$ or $I = I_0 e^{-\frac{R}{L}t}$ ------ (5)

Thus, the current decay exponentially from its maximum value $\left(I_0 = \frac{E}{R}\right)$ to zero

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putting $\frac{L}{R} = t$ (In eqn's)

we have

$$I = I_0 e^{-1} = \frac{I_0}{e} \quad \text{--- (6)}$$

$$= I_0 \left(\frac{1}{2.718}\right)$$

$$= 0.3679 I_0 \quad \text{--- (7)}$$

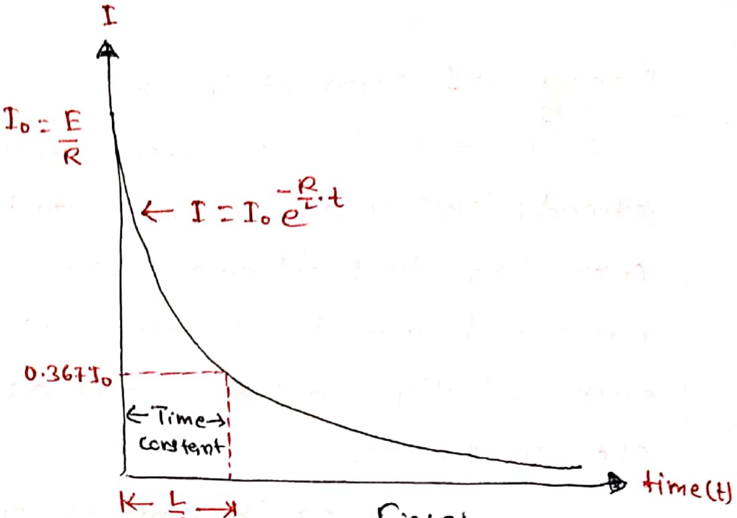

Fig.(3)

Time constant :

Time constant of LR circuit $= \frac{L}{R}$

The dimension of L are $\frac{\text{Volt}}{A/s} = \frac{V.s}{A}$

and dimension of R are $\frac{\text{Volt}}{\text{amp}}$

$\therefore$ Dimension of $\frac{L}{R} = \frac{\text{Volt sec}}{\text{amp}} \times \frac{\text{amp}}{\text{Volt}}$

$= \sec [T]$

The time constant of LR circuit may be defined as the time during which the current falls to $\frac{1}{e}$ of its maximum value

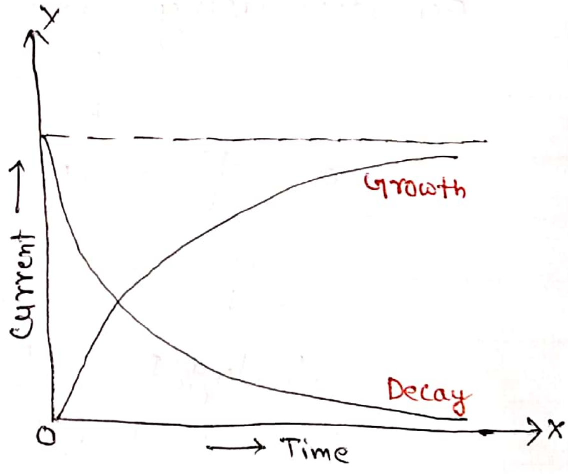

Fig.(4)

——— X ———