Gauss's Divergence Theorem:

If a closed surface $\vec{S}$ enclosed a volume v in a vecter field $\vec{A}$, then the serface integral of the normal component of $\vec{A}$ over the entire surface $\vec{S}$ is equal to the volume integral of divergence $\vec{A}$ over the volume v enclosed by the surface $\vec{S}$

or $$\iint_S \vec{A}\cdot\vec{ds} = \iiint_V (\text{div}\,\vec{A})\,dv$$

or $$\iint_S \vec{A}\cdot\vec{ds} = \iiint_V (\vec{\nabla}\cdot\vec{A})\,dv$$

Proof. Suppose a surface $\vec{S}$ enclosed a volume V in a field $\vec{A}$. Let the volume V be divided into large number of small elements of volume $\Delta V_1, \Delta V_2 --- \Delta V_i$ etc by enclosing surface area $\Delta\vec{S}_1, \Delta\vec{S}_2 ---- \Delta\vec{S}_i$ -- etc respectively. Taking the outward flux of the vecter field through a closed surface as positive and the inward flux as negative, we find that the outward flux from one face of a small element of volume say ΔV_1 will be equal and opposite to the outward flux from the common face of a neighbouring element say ΔV_2 and the two will cancel each other. As a result, the sum of all the outward fluxes from the elements of volume such as $\Delta V_1, \Delta V_2, --- \Delta V_i$ -- etc will be equal to the total outward flux from the whole volume V because all parts of the outward flux from the small elements of volume will cancel out except for those parts which are through the outer bounding surface $\vec{S}$

$\therefore$ Total outward flux through the closed surface $\vec{S}$ i.e the surface integral of the vecter $\vec{A}$ over the surface $\vec{S}$ is equal to the sum of the outward fluxes over the surface $\Delta\vec{S}_1, \Delta\vec{S}_2 --- \Delta\vec{S}_i$ etc

Teacher's Signature:

or $$\iint_S \vec{A}\cdot\vec{ds} = \sum \iint_{\Delta S_i} \vec{A}\cdot\vec{ds} \qquad \text{——} (1)$$

The value of div $\vec{A}$ in a small element like ΔV_i having a surface area ΔS_i is given by

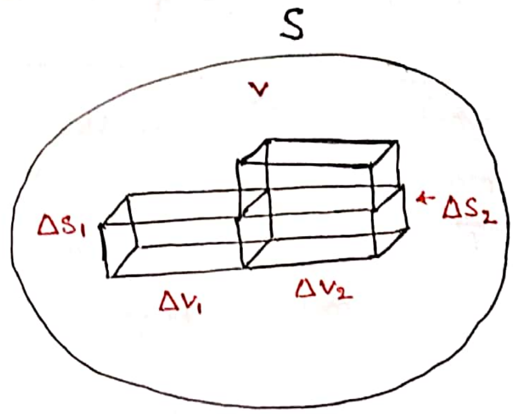

Fig.(1)

$$\text{div}\,\vec{A} = \lim_{\Delta V_i \to 0} \frac{1}{\Delta V_i} \iint_{\Delta S_i} \vec{A}\cdot\vec{ds}$$

or $$\iint_{\Delta S_i} \vec{A}\cdot\vec{ds} = (\text{div}\,\vec{A})'\,\Delta V_i$$

$$\therefore \sum \iint_{\Delta S_i} \vec{A}\cdot\vec{ds} = \sum (\text{div}\,\vec{A})\,\Delta V_i$$

where $\Delta V_i = dv$ i.e an infinite number of small volume elements are involved and $\Delta V_i \to 0$, we have

$$\sum (\text{div}\,\vec{A})\,\Delta V_i = \iiint_V (\text{div}\,\vec{A})\,\Delta V$$

$$\therefore \sum \iint_{\Delta S_i} \vec{A}\cdot\vec{ds} = \iiint_V (\text{div}\,\vec{A})\,\Delta V \qquad \text{——} (2)$$

Comparing (1) and (2), we get

$$\iint_S \vec{A}\cdot\vec{ds} = \iiint_V (\text{div}\,\vec{A})\,dv = \iiint_V (\vec{\nabla}\cdot\vec{A})\,dv$$

The importance of this theorem lies in the fact that it helps us to convert a surface integral into a volume integral and vice versa

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