*** PHYSICS ***

MATHEMATICAL PHYSICS

(L.S-03)

Dr. M.K. THAKUR
A.P.S.M. College, Barauni

STOKE'S THEOREM

The surface integral of the curl of vecter $\vec{A}$ over a a surface $\vec{s}$ of any shape is equal to the line integral of the vector field $\vec{A}$ over the boundary of that surface

or $$\iint_S (\text{curl}\vec{A})\cdot\vec{ds} = \oint \vec{A}\cdot\vec{dl}$$

Where $\oint$ represents the line integral over the closed path enclosing the surface $\vec{S}$.

Stoke's theorem may also be stated in the form the line integral of a vector field $\vec{A}$ around any closed curve C is equal to the surface integral of the curl $\vec{A}$ over an open surface S bounded by the curve $\vec{C}$. Mathematically

$$\oint_C \vec{A}\cdot\vec{dl} = \iint_S (\text{curl}\vec{A})\cdot\vec{ds} = \iint_S (\vec{\nabla}\times\vec{A})\cdot\vec{ds}$$

We shall prove the theorem given in the form

$$\iint_S (\text{curl}\vec{A})\cdot\vec{ds} = \oint \vec{A}\cdot\vec{dl}$$

Suppose a smooth closed curve l enclosed vector area $\vec{S}$ in vector field $\vec{A}$. Let the area $\vec{S}$ be divided into a large number of small area $\Delta S_1, \Delta S_2, \cdots \Delta S_i$ -- etc having perimeters $\Delta l_1, \Delta l_2, \cdots \Delta l_i$ etc respectively.

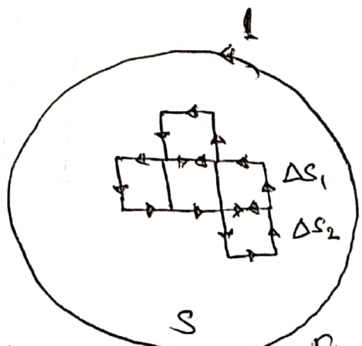

Fig. (1)

The line integrals of the vector $\vec{A}$ around each of the small path $\Delta l_1, \Delta l_2$ etc will be in the same sense. Therefore the line integral along the common boundary of two small area like ΔS_1 and ΔS_2 will cancel each other being in opposite direction.

Hence, the sum of all these line integrals will be equal to the line integral around l the boundary enclosing the whole area $\vec{S}$ because in traversing the small area all parts of the line integral will cancel out except for those parts which are along the outer boundary l. Thus the line integral around the closed curve l is equal to the sum of the line integral around the paths Δl_1, Δl_2, -- Δl_i etc

$$\oint \vec{A}\cdot\vec{dl} = \sum \oint \vec{A}\cdot\vec{dl} \qquad \text{--- (1)}$$

Any one of the small area ΔS_i will have a curl of the vector field of which the normal component

$$\text{Curl}_n \vec{A} = \lim_{\Delta S_i \to 0} \frac{1}{\Delta S_i} \oint \vec{A}\cdot\vec{dl}$$

or $$\oint \vec{A}\cdot\vec{dl} = (\text{Curl}_n \vec{A})\,\Delta S_i$$

$$\sum_{\Delta l_i} \oint \vec{A}\cdot\vec{dl} = \sum (\text{Curl}_n \vec{A})\,\Delta S_i$$

When $\Delta S_i = ds$ i.e an infinite number of small volume elements are involved and $\Delta S_i \to 0$ then

$$\sum (\text{Curl}_n \vec{A})\,\Delta S = \iint_S (\text{Curl}_n \vec{A})\cdot\vec{dS} = \iint_S \text{Curl}\,\vec{A}\cdot\vec{dS}$$

$$\therefore \sum_{\Delta l_i} \oint \vec{A}\cdot\vec{dl} = \iint_S \text{Curl}\,\vec{A}\cdot\vec{dS} \qquad \text{--- (2)}$$

From eqn (1) and (2), we get

$$\iint_S \text{Curl}\,\vec{A}\cdot\vec{dS} = \oint \vec{A}\cdot\vec{dl}$$

or $$\iint_S (\vec{\nabla}\times\vec{A})\cdot\vec{dS} = \oint_l \vec{A}\cdot\vec{dl}$$

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GREEN'S THEOREM If S is closed region in xy plane bounded by a simple closed curve C and ϕ and ψ are continuous function of x and y having continuous derivatives, then

$$\oint_C \phi\, dx + \psi\, dy = \iint_S \left(\frac{\partial \psi}{\partial x} - \frac{\partial \phi}{\partial y} \right) dx\, dy$$

where the curve C is traversed in the anticlockwise direction

Proof: Let S be a closed region in x-y plane bounded by a closed curve C. Suppose $\vec{A}$ is a vector field having ϕ and ψ as its x and y- components respectively, then

$$\vec{A} = \phi \hat{i} + \psi \hat{j} \quad \text{—— (1)}$$

In the x-y plane a displacement vector $\vec{dr}$ is given by

$$\vec{dr} = dx \hat{i} + dy \hat{j} \quad \text{—— (2)}$$

According to stoke's theorem

$$\oint_C \vec{A} \cdot \vec{dr} = \iint_S (\vec{\nabla} \times \vec{A}) \cdot \vec{ds} \quad \text{—— (3)}$$

From (1), (2) and (3)

$$\vec{A} \cdot \vec{dr} = (\phi \hat{i} + \psi \hat{j}) \cdot (dx \hat{i} + dy \hat{j}) = \phi\, dx + \psi\, dy \quad \text{—— (4)}$$

Also
$$\vec{\nabla} \times \vec{A} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \phi & \psi & 0 \end{vmatrix}$$

$$= \hat{i} \left| -\frac{\partial \psi}{\partial z} \right| + \hat{j} \left| \frac{\partial \phi}{\partial z} \right| + \hat{k} \left| \frac{\partial \psi}{\partial x} - \frac{\partial \phi}{\partial y} \right| \quad \text{—— (5)}$$

Consider a small area element $\vec{ds}$ on the surface S. As S lies in the x-y plane the area vector will point in the +z direction As the curve C is traversed in the anticlockwise direction

$$\therefore \vec{ds} = ds\, \hat{k} \quad \text{—— (6)}$$

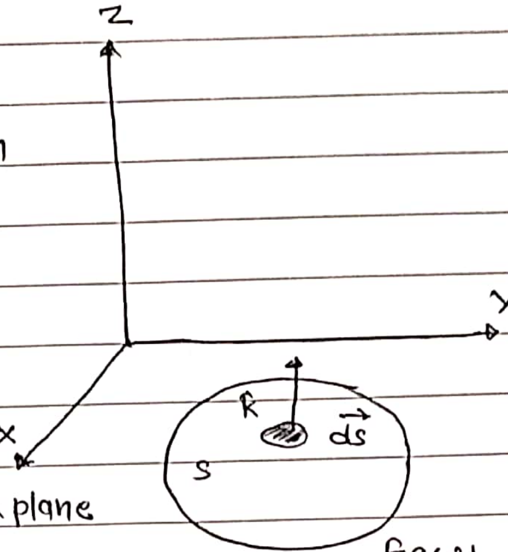
From (5) and (6)

$$(\vec{\nabla}\times\vec{A})\cdot\vec{ds} = \left[-\hat{i}\frac{\partial\psi}{\partial z} + \hat{j}\frac{\partial\phi}{\partial z} + \hat{k}\left(\frac{\partial\psi}{\partial x} - \frac{\partial\phi}{\partial y}\right)\right][ds\cdot\hat{k}]$$

$$= \left(\frac{\partial\psi}{\partial x} - \frac{\partial\phi}{\partial y}\right)ds \quad \text{—— (7)}$$

As $\hat{i}\cdot\hat{k} = \hat{j}\cdot\hat{k} = 0$ and $\hat{k}\cdot\hat{k} = 1$

Also $ds = dxdy$ — (8) as the area element ds lies in the x-y plane



Fig(.2)

Substituting the value of $\vec{A}\cdot\vec{dr}$ from (4), $(\vec{\nabla}\times\vec{A})\cdot\vec{ds}$ from (7) and ds from eqn (8) in Stoke's theorem (eqn (3)), we get

$$\oint_C \phi dx + \psi dy = \iint_S \left(\frac{\partial\psi}{\partial x} - \frac{\partial\phi}{\partial y}\right)ds$$

$$= \iint_S \left(\frac{\partial\psi}{\partial x} - \frac{\partial\phi}{\partial y}\right)dxdy$$

This eqn is known as Green's theorem in a plane

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