LECTURE SERIES-03

E.M. WAVE EQUATION IN A CONDUCTING MEDIUM

Let us consider a conducting medium of electrical conductivity σ and having finite permeability μ and permittivity ϵ. Suppose the medium is free from a charge source, hence charge density $\rho = 0$. In such a case, Maxwell's four field equations are:

$$\vec{\nabla}\cdot\vec{D} = \epsilon\vec{\nabla}\cdot\vec{E} = 0 \text{ or } \vec{\nabla}\cdot\vec{E} = 0 \quad \text{—— (1)}$$

$$\vec{\nabla}\cdot\vec{B} = 0 \quad \text{—— (2)}$$

$$\vec{\nabla}\times\vec{E} = -\frac{\partial\vec{B}}{\partial t} \quad \text{—— (3)}$$

$$\vec{\nabla}\times\vec{B} = \mu\left(\vec{J} + \epsilon\frac{\partial\vec{E}}{\partial t}\right) = \mu\left(\sigma\vec{E} + \epsilon\frac{\partial\vec{E}}{\partial t}\right) \quad \text{—— (4)}$$

As $\vec{J} = \sigma\vec{E}$

(1) For electric field $\vec{E}$: Let us eliminate $\vec{B}$ from eq^n (3) and (4). Taking curl of eq^n (3), we get

$$\vec{\nabla}\times(\vec{\nabla}\times\vec{E}) = \vec{\nabla}\times\left(-\frac{\partial\vec{B}}{\partial t}\right) = -\frac{\partial}{\partial t}(\vec{\nabla}\times\vec{B})$$

Substituting for $\vec{\nabla}\times\vec{B}$ from eq^n (4), we get

$$\vec{\nabla}\times(\vec{\nabla}\times\vec{E}) = -\frac{\partial}{\partial t}\left(\mu\sigma\vec{E} + \mu\epsilon\frac{\partial\vec{E}}{\partial t}\right)$$

$$= -\mu\sigma\frac{\partial\vec{E}}{\partial t} - \mu\epsilon\frac{\partial^2\vec{E}}{\partial t^2} \quad \text{—— (5)}$$

Applying the triple vector identity.

$$\vec{A}\times(\vec{B}\times\vec{C}) = \vec{B}(\vec{A}\cdot\vec{C}) - \vec{C}(\vec{A}\cdot\vec{B})$$

we get

$$\vec{\nabla}\times(\vec{\nabla}\times\vec{E}) = \vec{\nabla}(\vec{\nabla}\cdot\vec{E}) - \vec{E}(\vec{\nabla}\cdot\vec{\nabla})$$

$$= 0 - \vec{\nabla}^2\vec{E} \quad (\text{Since } \vec{\nabla}\cdot\vec{E} = 0)$$

$$= -\nabla^2\vec{E} \quad \text{—— (6)}$$

Substituting for $\vec{\nabla}\times(\vec{\nabla}\times\vec{E})$ in eq^n (5), we get

$$-\nabla^2\vec{E} = -\mu\sigma\frac{\partial\vec{E}}{\partial t} - \mu\epsilon\frac{\partial^2\vec{E}}{\partial t^2}$$

or $$\nabla^2\vec{E} = \mu\sigma\frac{\partial\vec{E}}{\partial t} + \mu\epsilon\frac{\partial^2\vec{E}}{\partial t^2} \quad \text{------} \quad (7)$$

This is the general wave equation for electric field $\vec{E}$. If the medium is non-conducting, $\sigma = 0$, then we have

$$\nabla^2\vec{E} = \mu\epsilon\frac{\partial^2\vec{E}}{\partial t^2} \quad \text{------} \quad (8)$$

Comparing eq^n (7) and (8), we find that $\left(\mu\sigma\frac{\partial\vec{E}}{\partial t}\right)$ is the dissipative terms,

(2) For magnetic field $\vec{B}$: Let us eliminate $\vec{E}$ from eq^n (3) and (4). Taking curl of eq^n (4), we get

$$\vec{\nabla}\times(\vec{\nabla}\times\vec{B}) = \vec{\nabla}\times\left(\mu\sigma\vec{E} + \mu\epsilon\frac{\partial\vec{E}}{\partial t}\right)$$

$$= \mu\sigma(\vec{\nabla}\times\vec{E}) + \mu\epsilon\frac{\partial}{\partial t}(\vec{\nabla}\times\vec{E})$$

$$= \mu\sigma\left(-\frac{\partial\vec{B}}{\partial t}\right) + \mu\epsilon\frac{\partial}{\partial t}\left(-\frac{\partial\vec{B}}{\partial t}\right) \quad (\text{From eq}^n\ (3))$$

$$= -\mu\sigma\frac{\partial\vec{B}}{\partial t} - \mu\epsilon\frac{\partial^2\vec{B}}{\partial t^2} \quad \text{------} \quad (9)$$

Applying the triple vector identity,

$$\vec{A}\times(\vec{B}\times\vec{C}) = \vec{B}(\vec{A}\cdot\vec{C}) - \vec{C}(\vec{A}\cdot\vec{B})$$

we get $$\vec{\nabla}\times(\vec{\nabla}\times\vec{B}) = \vec{\nabla}(\vec{\nabla}\cdot\vec{B}) - \vec{B}(\vec{\nabla}\cdot\vec{\nabla})$$

$$= 0 - \nabla^2\vec{B} = -\nabla^2\vec{B}$$

Substituting this in eq^n (9), we get

$$-\nabla^2 B = -\mu\sigma\frac{\partial\vec{B}}{\partial t} - \mu\epsilon\frac{\partial^2\vec{B}}{\partial t^2}$$

or $$\nabla^2\vec{B} = \mu\sigma\frac{\partial\vec{B}}{\partial t} + \mu\epsilon\frac{\partial^2\vec{B}}{\partial t^2} \quad \text{------} \quad (10)$$

This is the general wave equation for magnetic field vector $\vec{B}$

If the medium is non-conducting, $\sigma = 0$, then we have

$$\nabla^2 \vec{B} = \mu \epsilon \frac{\partial^2 \vec{B}}{\partial t^2} \quad \text{—— (11)}$$

Comparing eq^n (10) and (11), we find that $\left(\mu \sigma \frac{\partial \vec{B}}{\partial t}\right)$ is the dissipative term.

E.M. WAVE EQUATION IN FREE SPACE

For free space, Electrical conductivity $\sigma = 0$

charge density $\rho = 0$

Current density $\vec{J} = 0$

Relative permittivity (K i.e ϵ_r) $= 1$

Relative permeability $\mu_r = 1$

Hence, $\epsilon = \epsilon_0 \cdot \epsilon_r = \epsilon_0 \cdot 1 = \epsilon_0$

and $\mu = \mu_0 \cdot \mu_r = \mu_0 \cdot 1 = \mu_0$

In such a case, Maxwell's field equation are:

$$\vec{\nabla} \cdot \vec{D} = \epsilon_0 \vec{\nabla} \cdot \vec{E} = 0 \text{ or } \vec{\nabla} \cdot \vec{E} = 0 \quad \text{—— (1)}$$

$$\vec{\nabla} \cdot \vec{B} = 0 \quad \text{—— (2)}$$

$$\vec{\nabla} \times \vec{E} = -\frac{\partial \vec{B}}{\partial t} \quad \text{—— (3)}$$

$$\vec{\nabla} \times \vec{B} = \mu_0 \left(\vec{J} + \epsilon_0 \frac{\partial \vec{E}}{\partial t}\right) = \mu_0 \left(\sigma \vec{E} + \epsilon_0 \frac{\partial \vec{E}}{\partial t}\right)$$

$$= 0 + \mu_0 \epsilon_0 \frac{\partial \vec{E}}{\partial t} = \mu_0 \epsilon_0 \frac{\partial \vec{E}}{\partial t} \quad \text{—— (4)}$$

(Since $\sigma = 0$)

(i) For electric field $\vec{E}$: Let us eliminate $\vec{B}$ from eq^n (3) and (4). Taking Curl of eq^n (3), we have

$$\vec{\nabla} \times (\vec{\nabla} \times \vec{E}) = \vec{\nabla} \times \left(-\frac{\partial \vec{B}}{\partial t}\right) = -\frac{\partial}{\partial t}(\vec{\nabla} \times \vec{B})$$

Substituting for $(\vec{\nabla} \times \vec{B})$ from eq^n (4), we get

$$\vec{\nabla} \times (\vec{\nabla} \times \vec{E}) = -\frac{\partial}{\partial t}\left(\mu_0 \epsilon_0 \frac{\partial \vec{E}}{\partial t}\right) = -\mu_0 \epsilon_0 \frac{\partial^2 \vec{E}}{\partial t^2} \quad \text{—— (5)}$$

Teacher's Signature:

Applying the triple vector identity.

$$\vec{A}\times(\vec{B}\times\vec{C}) = \vec{B}(\vec{A}\cdot\vec{C}) - \vec{C}(\vec{A}\cdot\vec{B})$$

we get, $$\vec{\nabla}\times(\vec{\nabla}\times\vec{E}) = \vec{\nabla}(\vec{\nabla}\cdot\vec{E}) - \vec{E}(\vec{\nabla}\cdot\vec{\nabla})$$
$$= -\nabla^2\vec{E} \quad \text{—— (6)}$$

Since $\vec{\nabla}\cdot\vec{E} = 0$ (from eq^n (1))

From eq^n (5) and (6), we have

$$-\nabla^2\vec{E} = -\mu_0\epsilon_0\frac{\partial^2\vec{E}}{\partial t^2}$$

or $$\nabla^2\vec{E} = \mu_0\epsilon_0\frac{\partial^2\vec{E}}{\partial t^2} \quad \text{—— (7)}$$

This represents the wave equation for the vector field $\vec{E}$ in free space i.e in vacuum.

(2). For magnetic field $\vec{B}$ Let us eliminate $\vec{E}$ from eq^n (3) and (4). Taking curl of eq^n (4), we get

$$\vec{\nabla}\times(\vec{\nabla}\times\vec{B}) = \vec{\nabla}\times\left(\mu_0\epsilon_0\frac{\partial\vec{E}}{\partial t}\right) = \mu_0\epsilon_0\frac{\partial}{\partial t}(\vec{\nabla}\times\vec{E})$$

Substituting for $\vec{\nabla}\times\vec{E}$ from eq^n (3), we get

$$\vec{\nabla}\times(\vec{\nabla}\times\vec{B}) = \mu_0\epsilon_0\frac{\partial}{\partial t}\left(-\frac{\partial\vec{B}}{\partial t}\right) = -\mu_0\epsilon_0\frac{\partial^2\vec{B}}{\partial t^2} \quad \text{—— (8)}$$

Applying the triple vector identity.

$$\vec{A}\times(\vec{B}\times\vec{C}) = \vec{B}(\vec{A}\cdot\vec{C}) - \vec{C}(\vec{A}\cdot\vec{B})$$

we get $$\vec{\nabla}\times(\vec{\nabla}\times\vec{B}) = \vec{\nabla}(\vec{\nabla}\cdot\vec{B}) - \vec{B}(\vec{\nabla}\cdot\vec{\nabla})$$
$$= -\nabla^2\vec{B} \quad \text{—— (9)}$$

Since $\vec{\nabla}\cdot\vec{B} = 0$ (from eq^n (2))

From eq^n (8) and (9), we have

$$-\nabla^2\vec{B} = -\mu_0\epsilon_0\frac{\partial^2\vec{B}}{\partial t^2}$$

or $$\nabla^2\vec{B} = \mu_0\epsilon_0\frac{\partial^2\vec{B}}{\partial t^2} \quad \text{—— (10)}$$

Eq^n (7) and (10) the relation between the space and time variation of $\vec{E}$ and $\vec{B}$ are called three dimensional wave equation for $\vec{E}$ and $\vec{B}$ respectively.

The square root of the quantity that is the reciprocal of the co-efficient of the time derivative gives the phase velocity of the wave. These equation, therefore indicate that electromagnetic fields changing with time propagate in the form of electromagnetic waves with velocity $1/\sqrt{\mu_0 \epsilon_0}$.

Substituting the values of μ_0 and ϵ_0, we find that

$$\frac{1}{\sqrt{\mu_0 \epsilon_0}} = \frac{1}{\sqrt{(4\pi \times 10^{-7}\, Wb/A\cdot m^2)(8.9 \times 10^{-12}\, C^2/N\cdot m^2)}} = 3.0 \times 10^{8}\, m/s$$

Thus $\frac{1}{\sqrt{\mu_0 \epsilon_0}} = c$ (the velocity of light)

Maxwell's predicted that electromagnetic disturbance propagates in free space with a speed equal to that of light and hence light waves were confirmed to be electromagnetic in nature

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